重積分　解答

1　ここの問題はいずれも長方形領域

$$D = \{(x, y) \mid a \leqq x \leqq b,\ c \leqq y \leqq d\}$$

における重積分なので，累次積分の公式

$$\begin{aligned}\iint_D f(x,y)\,dx\,dy &= \int_a^b dx \int_c^d f(x,y)\,dy \\ &= \int_c^d dy \int_a^b f(x,y)\,dx\end{aligned}$$

を用いればよい。

(1) $\displaystyle\iint_D xy^3\,dx\,dy, \quad D = \{(x,y) \mid 0 \leqq x \leqq 2,\ 0 \leqq y \leqq 1\}$

$$\begin{aligned}\iint_D xy^3\,dx\,dy &= \int_0^2 dx \int_0^1 xy^3\,dy = \int_0^2 dx \left[\frac{xy^4}{4}\right]_{y=0}^{y=1} = \int_0^2 \left(\frac{x}{4} - 0\right)\,dx \\ &= \left[\frac{x^2}{8}\right]_0^2 = \frac{4}{8} - 0 = \frac{1}{2}\end{aligned}$$

(2) $\displaystyle\iint_D (2x + 3y - 1)\,dx\,dy, \quad D = \{(x,y) \mid 1 \leqq x \leqq 3,\ 1 \leqq y \leqq 4\}$

$$\begin{aligned}\iint_D (2x+3y-1)\,dx\,dy &= \int_1^3 dx \int_1^4 (2x+3y-1)\,dy = \int_1^3 dx \left[2xy + \frac{3}{2}y^2 - y\right]_{y=1}^{y=4} \\ &= \int_1^3 \left(8x + 24 - 4 - \left(2x + \frac{3}{2} - 1\right)\right)\,dx \\ &= \int_1^3 \left(6x + \frac{39}{2}\right)\,dx = \left[3x^2 + \frac{39}{2}x\right]_1^3 = 63\end{aligned}$$

(3) $\displaystyle\iint_D (x^2 - y^2)\,dx\,dy, \quad D = \{(x,y) \mid -1 \leqq x \leqq 1,\ 0 \leqq y \leqq 1\}$

$$\begin{aligned}\iint_D (x^2 - y^2)\,dx\,dy &= \int_{-1}^1 dx \int_0^1 (x^2 - y^2)\,dy = \int_{-1}^1 dx \left[x^2 y - \frac{y^3}{3}\right]_{y=0}^{y=1} \\ &= \int_{-1}^1 \left(x^2 - \frac{1}{3}\right)\,dx = \left[\frac{x^3}{3} - \frac{1}{3}x\right]_{-1}^1 = 0\end{aligned}$$

(4) $\displaystyle\iint_D (x^2 - 2xy + x + 3y)\,dx\,dy, \quad D = \{(x,y) \mid 1 \leqq x \leqq 2,\ 2 \leqq y \leqq 4\}$

$$
\begin{aligned}
\iint_D (x^2-2xy+x+3y)\,dx\,dy &= \int_1^2 dx \int_2^4 (x^2-2xy+x+3y)\,dy \\
&= \int_1^2 dx \left[x^2y - xy^2 + xy + \frac{3}{2}y^2\right]_{y=2}^{y=4} \\
&= \int_1^2 (4x^2-16x+4x+24-(2x^2-4x+2x+6))\,dx \\
&= \int_1^2 (2x^2-10x+18)\,dx \\
&= \left[\frac{2}{3}x^3-5x^2+18x\right]_1^2 = \frac{23}{3}
\end{aligned}
$$

(5) $\displaystyle\iint_D \frac{dx\,dy}{x+y}, \quad D=\{(x,y)\mid 1\leqq x\leqq 2,\ 1\leqq y\leqq 2\}$

$$
\begin{aligned}
\iint_D \frac{dx\,dy}{x+y} &= \int_1^2 dx \int_1^2 \frac{dy}{x+y} = \int_1^2 dx\big[\log(x+y)\big]_{y=1}^{y=2} \\
&= \int_1^2 (\log(x+2)-\log(x+1))\,dx \\
&= \big[(x+2)\log(x+2)-x-((x+1)\log(x+1)-x)\big]_1^2 \\
&= 4\log 4 - 3\log 3 - (3\log 3 - 2\log 2) = 10\log 2 - 6\log 3
\end{aligned}
$$

(6) $\displaystyle\iint_D \frac{dx\,dy}{2x+y+1}, \quad D=\{(x,y)\mid 0\leqq x\leqq 1,\ 0\leqq y\leqq 4\}$

$$
\begin{aligned}
\iint_D \frac{dx\,dy}{2x+y+1} &= \int_0^1 dx \int_0^4 \frac{dy}{2x+y+1} = \int_0^1 dx\big[\log(2x+y+1)\big]_{y=0}^{y=4} \\
&= \int_0^1 (\log(2x+5)-\log(2x+1))\,dx \\
&= \left[\frac{2x+5}{2}\log(2x+5)-x-\left(\frac{2x+1}{2}\log(2x+1)-x\right)\right]_0^1 \\
&= \frac{7}{2}\log 7 - \frac{3}{2}\log 3 - \left(\frac{5}{2}\log 5 - \frac{1}{2}\log 1\right) \\
&= \frac{7}{2}\log 7 - \frac{3}{2}\log 3 - \frac{5}{2}\log 5
\end{aligned}
$$

(7) $\displaystyle\iint_D \frac{dx\,dy}{(x+2y)^2}, \quad D=\{(x,y)\mid 1\leqq x\leqq 2,\ 0\leqq y\leqq 1\}$

$$
\begin{aligned}
\iint_D \frac{dx\,dy}{(x+2y)^2} &= \int_1^2 dx \int_0^1 \frac{dy}{(x+2y)^2} = \int_1^2 dx\left[-\frac{1}{2}\cdot\frac{1}{x+2y}\right]_{y=0}^{y=1} \\
&= \int_1^2 \left(-\frac{1}{2(x+2)}+\frac{1}{2x}\right)dx = \left[-\frac{1}{2}\log(x+2)+\frac{1}{2}\log x\right]_1^2 \\
&= -\frac{1}{2}\log 4 + \frac{1}{2}\log 2 + \frac{1}{2}\log 3 = \frac{\log 3 - \log 2}{2}
\end{aligned}
$$

(8) $\displaystyle \iint_D \frac{dx\,dy}{(2x+3y)^3}, \quad D = \{(x,y) \mid 1 \leqq x \leqq 3,\ 1 \leqq y \leqq 2\}$

$$\begin{aligned}
\iint_D \frac{dx\,dy}{(2x+3y)^3} &= \int_1^3 dx \int_1^2 \frac{dy}{(2x+3y)^3} = \int_1^3 \left[-\frac{1}{6}\cdot\frac{1}{(2x+3y)^2}\right]_{y=1}^{y=2} \\
&= -\frac{1}{6}\int_1^3 \left(\frac{1}{(2x+6)^2} - \frac{1}{(2x+3)^2}\right) dx \\
&= -\frac{1}{6}\left[-\frac{1}{2}\cdot\frac{1}{2x+6} + \frac{1}{2}\cdot\frac{1}{2x+3}\right]_1^3 \\
&= \frac{1}{12}\left(\frac{1}{12} - \frac{1}{9} - \frac{1}{8} + \frac{1}{5}\right) = \frac{17}{4320}
\end{aligned}$$

(9) $\displaystyle \iint_D \frac{2x}{(x+y)^2}\,dx\,dy, \quad D = \{(x,y) \mid 1 \leqq x \leqq 2,\ 2 \leqq y \leqq 3\}$

$$\begin{aligned}
\iint_D \frac{2x}{(x+y)^2}\,dx\,dy &= \int_1^2 dx \int_2^3 \frac{2x}{(x+y)^2}\,dy = \int_1^2 dx \left[-\frac{2x}{x+y}\right]_{y=2}^{y=3} \\
&= \int_1^2 \left(-\frac{2x}{x+3} + \frac{2x}{x+2}\right) dx = \int_1^2 \left(\frac{6}{x+3} - \frac{4}{x+2}\right) dx \\
&= \big[6\log(x+3) - 4\log(x+2)\big]_1^2 = 6\log 5 - 4\log 4 - (6\log 4 - 4\log 3) \\
&= 6\log 5 - 20\log 2 + 4\log 3
\end{aligned}$$

(10) $\displaystyle \iint_D \frac{xy^2}{(x+y)^3}\,dx\,dy, \quad D = \{(x,y) \mid 1 \leqq x \leqq 2,\ 2 \leqq y \leqq 3\}$

$$\begin{aligned}
\iint_D \frac{xy^2}{(x+y)^3}\,dx\,dy &= \int_2^3 dy \int_1^2 \frac{xy^2}{(x+y)^3}\,dx = \int_2^3 dy \int_1^2 \frac{(x+y-y)y^2}{(x+y)^3}\,dx \\
&= \int_2^3 dy \int_1^2 \left(\frac{y^2}{(x+y)^2} - \frac{y^3}{(x+y)^3}\right) dx \\
&= \int_2^3 dy \left[-\frac{y^2}{x+y} + \frac{1}{2}\cdot\frac{y^3}{(x+y)^2}\right]_{x=1}^{x=2} \\
&= \int_2^3 \left(-\frac{y^2}{y+2} + \frac{1}{2}\cdot\frac{y^3}{(y+2)^2} + \frac{y^2}{y+1} - \frac{1}{2}\cdot\frac{y^3}{(y+1)^2}\right) dy = (\#)
\end{aligned}$$

ここで分子に現れる y^2, y^3 を次のように書き換える。

$$\begin{aligned}
y^2 &= (y+2-2)^2 = (y+2)^2 - 4(y+2) + 4 \\
y^3 &= (y+2-2)^3 = (y+2)^3 - 6(y+2)^2 + 12(y+2) - 8 \\
y^2 &= (y+1-1)^2 = (y+1)^2 - 2(y+1) + 1 \\
y^3 &= (y+1-1)^3 = (y+1)^3 - 3(y+1)^2 + 3(y+1) - 1
\end{aligned}$$

これらを代入すると

$$\begin{aligned}(\#) &= \int_2^3 \left(\frac{2}{y+2} - \frac{4}{(y+2)^2} - \frac{1}{2} \cdot \frac{1}{y+1} + \frac{1}{2} \cdot \frac{1}{(y+1)^2} \right) dy \\ &= \left[2\log(y+2) + \frac{4}{y+2} - \frac{1}{2}\log(y+1) - \frac{1}{2} \cdot \frac{1}{y+1} \right]_2^3 \\ &= -\frac{19}{120} + 2\log 5 + \frac{1}{2}\log 3 - 5\log 2\end{aligned}$$

(11) $\displaystyle\iint_D e^{3x-y}\,dx\,dy, \quad D = \{(x,y) \mid 0 \leqq x \leqq 2,\ -1 \leqq y \leqq 0\}$

$$\begin{aligned}\iint_D e^{3x-y}\,dx\,dy &= \int_0^2 dx \int_{-1}^0 e^{3x-y}\,dy = \int_0^2 dx \big[-e^{3x-y}\big]_{y=-1}^{y=0} \\ &= \int_0^2 (-e^{3x} + e^{3x+1})\,dx = \left[-\frac{e^{3x}}{3} + \frac{e^{3x+1}}{3} \right]_0^2 = \frac{e^7 - e^6 - e + 1}{3}\end{aligned}$$

(12) $\displaystyle\iint_D e^{2x+y-1}\,dx\,dy, \quad D = \{(x,y) \mid 0 \leqq x \leqq 3,\ 0 \leqq y \leqq 2\}$

$$\begin{aligned}\iint_D e^{2x+y-1}\,dx\,dy &= \int_0^3 dx \int_0^2 e^{2x+y-1}\,dy = \int_0^3 dx \big[e^{2x+y-1}\big]_{y=0}^{y=2} \\ &= \int_0^3 (e^{2x+1} - e^{2x-1})\,dx = \left[\frac{e^{2x+1}}{2} - \frac{e^{2x-1}}{2} \right]_0^3 = \frac{e^7 - e^5 - e + e^{-1}}{2}\end{aligned}$$

(13) $\displaystyle\iint_D xe^{xy}\,dx\,dy, \quad D = \{(x,y) \mid 0 \leqq x \leqq 2,\ 0 \leqq y \leqq 1\}$

$$\begin{aligned}\iint_D xe^{xy}\,dx\,dy &= \int_0^2 dx \int_0^1 xe^{xy}\,dy = \int_0^2 \big[e^{xy}\big]_{y=0}^{y=1} = \int_0^2 (e^x - 1)\,dx \\ &= \big[e^x - x\big]_0^2 = e^2 - 3\end{aligned}$$

(14) $\displaystyle\iint_D ye^{xy}\,dx\,dy, \quad D = \{(x,y) \mid 0 \leqq x \leqq 2,\ 0 \leqq y \leqq 1\}$

$$\begin{aligned}\iint_D ye^{xy}\,dx\,dy &= \int_0^1 dy \int_0^2 ye^{xy}\,dx = \int_0^1 \big[e^{xy}\big]_{x=0}^{x=2} = \int_0^1 (e^{2y} - 1)\,dy \\ &= \left[\frac{e^{2y}}{2} - y \right]_0^1 = \frac{e^2 - 3}{2}\end{aligned}$$

注意　(13), (14) では累次積分する順番が大事である。

(15) $\displaystyle\iint_D \cos(x-3y)\,dx\,dy, \quad D = \left\{(x,y) \mid 0 \leqq x \leqq \frac{\pi}{4},\ \frac{\pi}{4} \leqq y \leqq \frac{\pi}{2}\right\}$

$$
\begin{aligned}
\iint_D \cos(x-3y)\,dx\,dy &= \int_0^{\frac{\pi}{4}} dx \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \cos(x-3y)\,dy = \int_0^{\frac{\pi}{4}} dx \left[-\frac{1}{3}\sin(x-3y)\right]_{y=\frac{\pi}{4}}^{y=\frac{\pi}{2}} \\
&= -\frac{1}{3}\int_0^{\frac{\pi}{4}} \left(\sin\left(x-\frac{3}{2}\pi\right) - \sin\left(x-\frac{3}{4}\pi\right)\right) dx \\
&= -\frac{1}{3}\left[-\cos\left(x-\frac{3}{2}\pi\right) + \cos\left(x-\frac{3}{4}\pi\right)\right]_0^{\frac{\pi}{4}} \\
&= \frac{1}{3}\left(\cos\left(-\frac{5}{4}\pi\right) - \cos\left(-\frac{\pi}{2}\right) - \cos\left(-\frac{3}{2}\pi\right) + \cos\left(-\frac{3}{4}\pi\right)\right) \\
&= -\frac{\sqrt{2}}{3}
\end{aligned}
$$

(16) $\displaystyle\iint_D \sin(3x+2y)\,dx\,dy, \quad D = \left\{(x,y) \mid 0 \leqq x \leqq \frac{\pi}{12},\ 0 \leqq y \leqq \frac{\pi}{4}\right\}$

$$
\begin{aligned}
\iint_D \sin(3x+2y)\,dx\,dy &= \int_0^{\frac{\pi}{12}} dx \int_0^{\frac{\pi}{4}} \sin(3x+2y)\,dy = \int_0^{\frac{\pi}{12}} dx \int_0^{\frac{\pi}{4}} \left[-\frac{1}{2}\cos(3x+2y)\right]_{y=0}^{y=\frac{\pi}{4}} \\
&= -\frac{1}{2}\int_0^{\frac{\pi}{12}} \left(\cos\left(3x+\frac{\pi}{2}\right) - \cos 3x\right) dx \\
&= -\frac{1}{2}\left[\frac{1}{3}\sin\left(3x+\frac{\pi}{2}\right) - \frac{1}{3}\sin 3x\right]_0^{\frac{\pi}{12}} \\
&= -\frac{1}{6}\left(\sin\frac{3}{4}\pi - \sin\frac{\pi}{4} - \sin\frac{\pi}{2}\right) = \frac{1}{6}
\end{aligned}
$$

(17) $\displaystyle\iint_D \log(x+y)\,dx\,dy, \quad D = \{(x,y) \mid 0 \leqq x \leqq 1,\ 1 \leqq y \leqq 2\}$

$$
\begin{aligned}
\iint_D \log(x+y)\,dx\,dy &= \int_0^1 dx \int_1^2 \log(x+y)\,dy = \int_0^1 \big[(x+y)\log(x+y) - y\big]_{y=1}^{y=2} \\
&= \int_0^1 ((x+2)\log(x+2) - 2 - (x+1)\log(x+1) + 1)\,dx \\
&= \left[\frac{(x+2)^2}{2}\log(x+2) - \frac{(x+2)^2}{4} - \frac{(x+1)^2}{2}\log(x+1) + \frac{(x+1)^2}{4} - x\right]_0^1 \\
&= -\frac{3}{2} + \frac{9}{2}\log 3 - 4\log 2
\end{aligned}
$$

(18) $\displaystyle\iint_D \log(2x+y+1)\,dx\,dy, \quad D = \{(x,y) \mid 0 \leqq x \leqq 1,\ 0 \leqq y \leqq 2\}$

$$
\begin{aligned}
\iint_D \log(2x+y+1)\,dx\,dy &= \int_0^1 dx \int_0^2 \log(2x+y+1)\,dy \\
&= \int_0^1 dx \big[(2x+y+1)\log(2x+y+1)-y\big]_{y=0}^{y=2} \\
&= \int_0^1 ((2x+3)\log(2x+3)-2-(2x+1)\log(2x+1))\,dx \\
&= \Big[\frac{(2x+3)^2}{4}\log(2x+3)-\frac{(2x+3)^2}{8} \\
&\qquad -\frac{(2x+1)^2}{4}\log(2x+1)+\frac{(2x+1)^2}{8}-2x\Big]_0^1 \\
&= -3+\frac{25}{4}\log 5-\frac{9}{2}\log 3
\end{aligned}
$$

2 与えられた領域 D を縦線集合あるいは横線集合として表して，その表示を用いて累次積分に持ち込んで求める。縦線集合とは

$$D = \{(x,y) \mid a \leqq x \leqq b,\ \varphi_1(x) \leqq y \leqq \varphi_2(x)\}$$

というように表される集合で，この場合には

$$\iint_D f(x,y)\,dx\,dy = \int_a^b dx \int_{\varphi_1(x)}^{\varphi_2(x)} f(x,y)\,dy$$

により累次積分ができる。横線集合とは

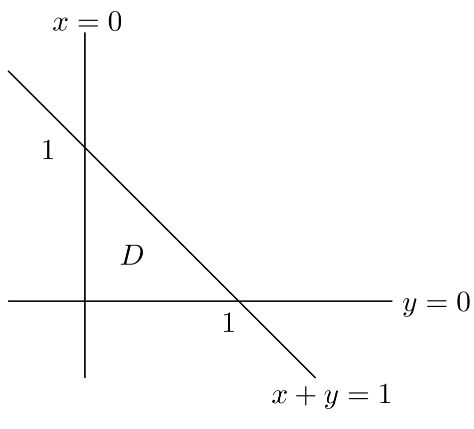
$$D = \{(x,y) \mid c \leqq y \leqq d,\ \psi_1(y) \leqq x \leqq \psi_2(y)\}$$

というように表される集合で，この場合には

$$\iint_D f(x,y)\,dx\,dy = \int_c^d dy \int_{\psi_1(y)}^{\psi_2(y)} f(x,y)\,dx$$

により累次積分ができる。

(1) $\displaystyle\iint_D \frac{y}{1+x}\,dx\,dy$，$D$: 直線 $x=0, y=0, x+y=1$ で囲まれる領域



領域 D は縦線集合として

$$D = \{(x, y) \mid 0 \leqq x \leqq 1,\ 0 \leqq y \leqq 1 - x\}$$

と表すことができる。この表示を用いて重積分を累次積分に書き換えて求める。

$$\iint_D \frac{y}{1+x}\,dx\,dy = \int_0^1 dx \int_0^{1-x} \frac{y}{1+x}\,dy = \int_0^1 dx \left[\frac{y^2}{2(1+x)}\right]_{y=0}^{y=1-x} = \int_0^1 \frac{(1-x)^2}{2(1+x)}\,dx$$
$$= \int_0^1 \left(\frac{x}{2} - \frac{3}{2} + \frac{2}{1+x}\right) dx = 2\log 2 - \frac{5}{4}$$

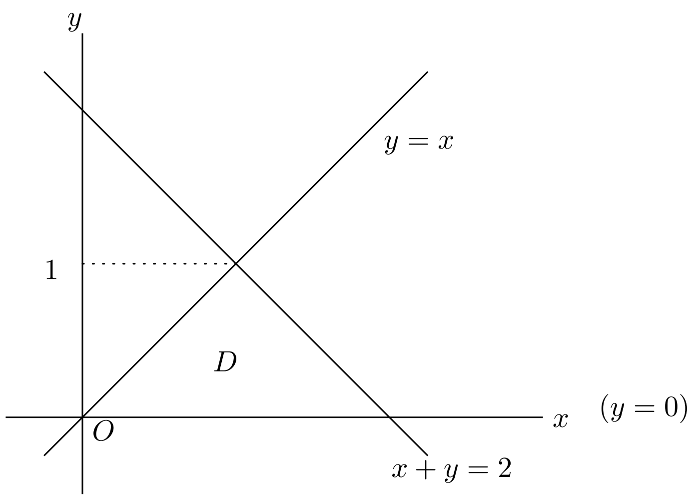
(2) $\displaystyle\iint_D \frac{x}{1+y}\,dx\,dy$, D : 直線 $x = 0, y = 0, x + y = 1$ で囲まれる領域

領域 D は前問と同じだが，今度は横線集合として表すとあとの計算が楽になる。

$$D = \{(x, y) \mid 0 \leqq y \leqq 1,\ 0 \leqq x \leqq 1 - y\}$$

$$\iint_D \frac{x}{1+y}\,dx\,dy = \int_0^1 dy \int_0^{1-y} \frac{x}{1+y}\,dx = \int_0^1 dy \left[\frac{x^2}{2(1+y)}\right]_{x=0}^{x=1-y} = \int_0^1 \frac{(1-y)^2}{2(1+y)}\,dy$$
$$= 2\log 2 - \frac{5}{4}$$

(3) $\displaystyle\iint_D e^{x+y}\,dx\,dy$, D : 直線 $y = 0, x = y, x + y = 2$ で囲まれる領域

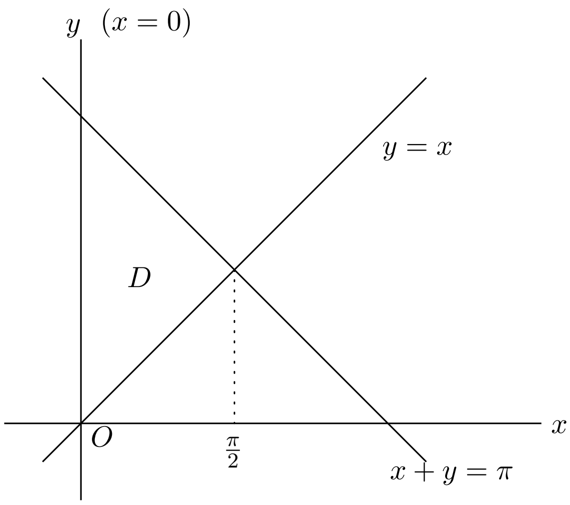


この領域は横線集合として表す方がが表しやすい。

$$D = \{(x, y) \mid 0 \leqq y \leqq 1,\ y \leqq x \leqq 2 - y\}$$

$$\iint_D e^{x+y}\,dx\,dy = \int_0^1 dy \int_y^{2-y} e^{x+y}\,dx = \int_0^1 dy \big[e^{x+y}\big]_{x=y}^{x=2-y} = \int_0^1 (e^2 - e^{2y})\,dy$$
$$= \left[e^2 y - \frac{1}{2}e^{2y}\right]_0^1 = \frac{e^2+1}{2}$$

(4) $\displaystyle\iint_D \cos(x+y)\,dx\,dy$, D : 直線 $x=0, x=y, x+y=\pi$で囲まれる領域

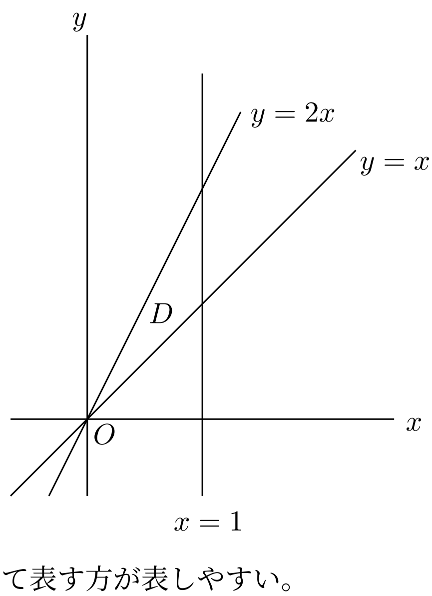


この領域は縦線集合として表す方が表しやすい。

$$D=\left\{(x,y)\mid 0\leqq x\leqq \frac{\pi}{2},\, x\leqq y\leqq \pi-x\right\}$$

$$\iint_D \cos(x+y)\,dx\,dy=\int_0^{\frac{\pi}{2}}dx\int_x^{\pi-x}\cos(x+y)\,dy=\int_0^{\frac{\pi}{2}}dx\Big[\sin(x+y)\Big]_{y=x}^{y=\pi-x}$$
$$=\int_0^{\frac{\pi}{2}}(\sin\pi-\sin 2x)\,dx=\left[\frac{1}{2}\cos 2x\right]_0^{\frac{\pi}{2}}=-1$$

(5) $\displaystyle\iint_D \frac{y}{1+x^2}\,dx\,dy$, D : 直線 $y=x, y=2x, x=1$ で囲まれる領域

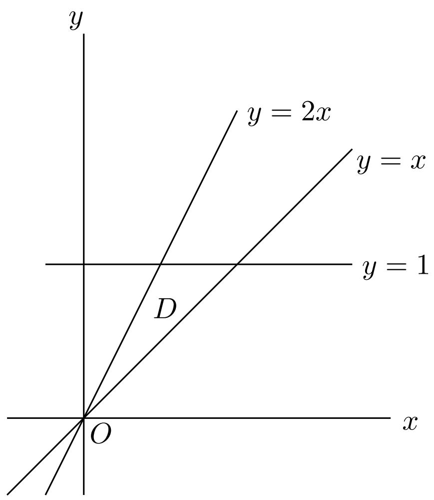


この領域は縦線集合として表す方が表しやすい。

$$D=\{(x,y)\mid 0\leqq x\leqq 1,\, x\leqq y\leqq 2x\}$$

$$\iint_D \frac{y}{1+x^2}\,dx\,dy = \int_0^1 dx \int_x^{2x} \frac{y}{1+x^2}\,dy = \int_0^1 dx \left[\frac{y^2}{2(1+x^2)}\right]_{y=x}^{y=2x} = \int_0^1 \frac{3x^2}{2(1+x^2)}\,dx$$
$$= \frac{3}{2}\int_0^1 \left(1 - \frac{1}{1+x^2}\right) dx = \frac{3}{2}[x - \tan^{-1} x]_0^1 = \frac{3}{2}\left(1 - \frac{\pi}{4}\right)$$

(6) $\displaystyle\iint_D ye^x\,dx\,dy,\ \ D$: 直線 $y = x, y = 2x, y = 1$ で囲まれる領域

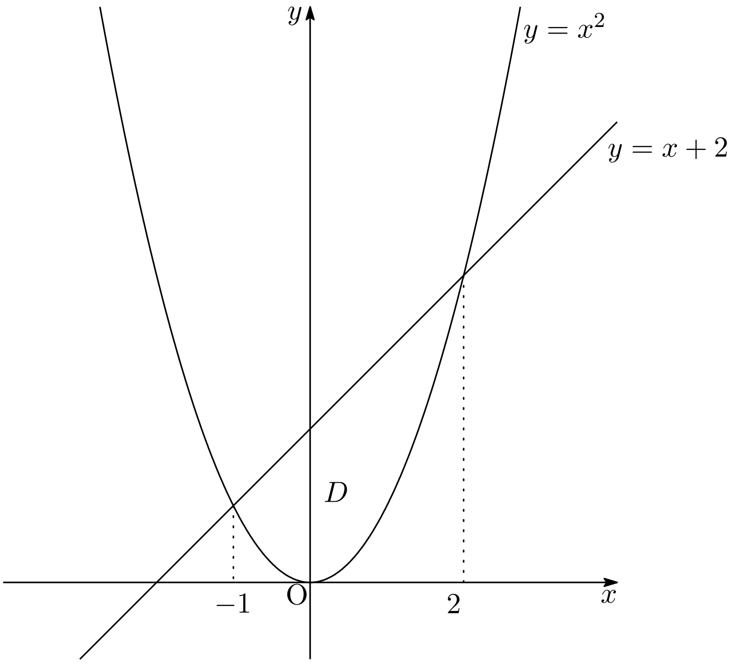


この領域は横線集合として表す方がが表しやすい。

$$D = \left\{(x, y) \mid 0 \leqq y \leqq 1,\ \frac{y}{2} \leqq x \leqq y\right\}$$

$$\iint_D ye^x\,dx\,dy = \int_0^1 dy \int_{\frac{y}{2}}^{y} ye^x\,dx = \int_0^1 dy\big[ye^x\big]_{x=\frac{y}{2}}^{x=y} = \int_0^1 \left(ye^y - ye^{\frac{y}{2}}\right) dy$$
$$= \big[ye^y\big]_0^1 - \int_0^1 e^y\,dy - \left(\left[2ye^{\frac{y}{2}}\right]_0^1 - \int_0^1 2e^{\frac{y}{2}}\,dy\right) = 2\sqrt{e} - 3$$

(7) $\displaystyle\iint_D (y - x^2)\,dx\,dy,\ \ D$: 曲線 $y = x^2$ と直線 $y = x + 2$ で囲まれる領域



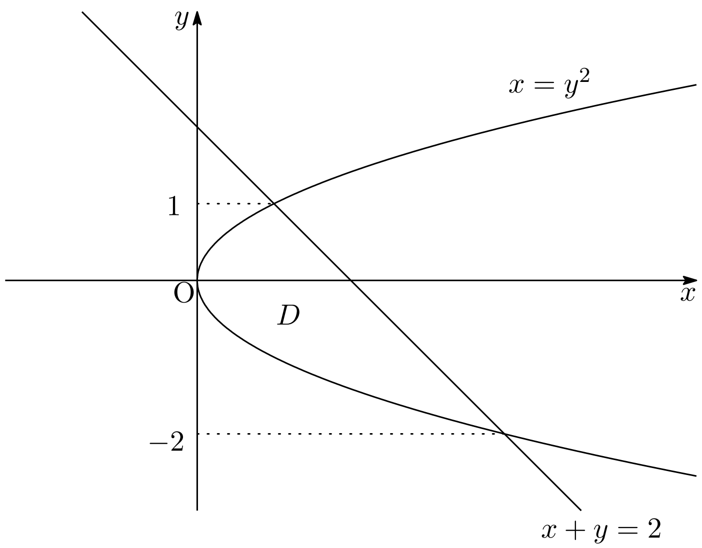
交点の x 座標は，$x^2 = x + 2$ を解いて $x = -1, 2$ となる。

この領域は縦線集合として表す方が表しやすい。

$$D = \{(x, y) \mid -1 \leqq x \leqq 2,\ x^2 \leqq y \leqq x + 2\}$$

$$\begin{aligned}\iint_D (y - x^2)\,dx\,dy &= \int_{-1}^{2} dx \int_{x^2}^{x+2} (y - x^2)\,dy = \int_{-1}^{2} \left[\frac{y^2}{2} - x^2 y\right]_{y=x^2}^{y=x+2} \\ &= \int_{-1}^{2} \left(\frac{(x+2)^2}{2} - x^2(x+2) - \frac{x^4}{2} + x^4\right) dx \\ &= \int_{-1}^{2} \left(\frac{x^4}{2} - x^3 - \frac{3}{2}x^2 + 2x + 2\right) dx = \frac{81}{20}\end{aligned}$$

(8) $\displaystyle\iint_D (x - y)\,dx\,dy$，$D$: 曲線 $x = y^2$ と直線 $x + y = 2$ で囲まれる領域



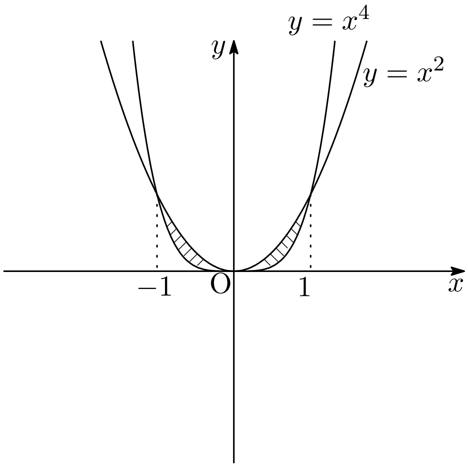
交点の y 座標は，$y^2 = 2 - y$ を解いて $y = -2, 1$ となる。

この領域は横線集合として表す方がが表しやすい。

$$D = \{(x, y) \mid -2 \leqq y \leqq 1,\ y^2 \leqq x \leqq 2 - y\}$$

$$\begin{aligned}\iint_D (x - y)\,dx\,dy &= \int_{-2}^{1} dy \int_{y^2}^{2-y} (x - y)\,dx = \int_{-2}^{1} dy \left[\frac{x^2}{2} - yx\right]_{x=y^2}^{x=2-y} \\ &= \int_{-2}^{1} \left(\frac{(2-y)^2}{2} - y(2-y) - \frac{y^4}{2} + y^3\right) dy \\ &= \int_{-2}^{1} \left(-\frac{y^4}{2} + y^3 + \frac{3}{2}y^2 - 4y + 2\right) dy = \frac{189}{20}\end{aligned}$$

(9) $\displaystyle\iint_D (x^2 + xy)\,dx\,dy$，$D$: 曲線 $y = x^2$ と $y = x^4$ で囲まれる領域



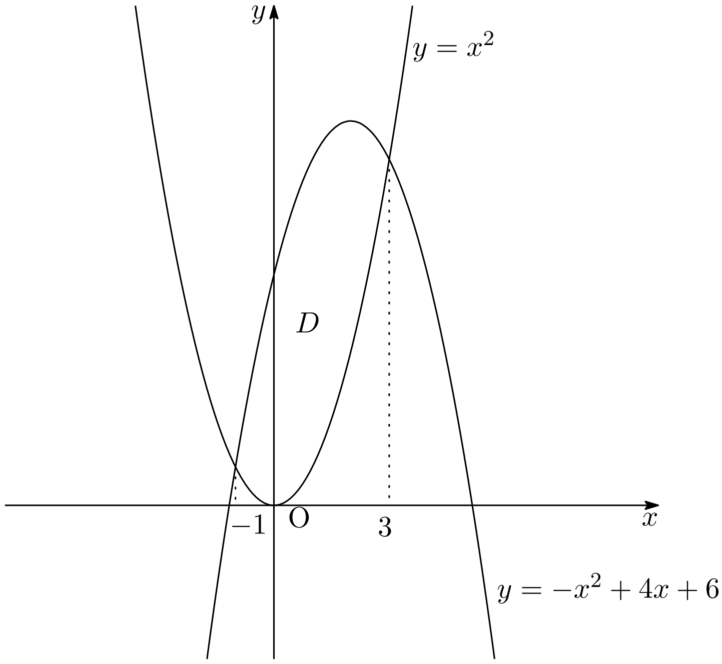
交点の x 座標は，$x^2 = x^4$ を解いて $x = 0, \pm 1$ となる。

この領域は縦線集合として表す方が表しやすい。

$$D = \{(x, y) \mid -1 \leqq x \leqq 1,\ x^4 \leqq y \leqq x^2\}$$

$$\begin{aligned}\iint_D (x^2 + xy)\, dx\, dy &= \int_{-1}^{1} dx \int_{x^4}^{x^2} (x^2 + xy)\, dy = \int_{-1}^{1} dx \left[x^2 y + x \cdot \frac{y^2}{2}\right]_{y=x^4}^{y=x^2} \\ &= \int_{-1}^{1} \left(x^4 + \frac{x^5}{2} - x^6 - \frac{x^9}{2}\right) dx = \frac{4}{35}\end{aligned}$$

(10) $\displaystyle\iint_D xy\, dx\, dy$, D : 曲線 $y = x^2, y = -x^2 + 4x + 6$ で囲まれる領域



交点の x 座標は，$x^2 = -x^2 + 4x + 6$ を解いて $x = -1, 3$ となる。

この領域は縦線集合として表す方が表しやすい。

$$D = \{(x, y) \mid -1 \leqq x \leqq 3,\ x^2 \leqq y \leqq -x^2 + 4x + 6\}$$

$$\begin{aligned}\iint_D xy\, dx\, dy &= \int_{-1}^{3} dx \int_{x^2}^{-x^2+4x+6} xy\, dy = \int_{-1}^{3} \left[x \cdot \frac{y^2}{2}\right]_{x^2}^{-x^2+4x+6} \\ &= \int_{-1}^{3} \left(x \cdot \frac{(-x^2 + 4x + 6)^2}{2} - x \cdot \frac{x^4}{2}\right) dx \\ &= \int_{-1}^{3} (-4x^4 + 2x^3 + 24x^2 + 18x)\, dx = \frac{704}{5}\end{aligned}$$