§5 高次偏導関数，微分の順序交換，Taylor の定理 演習問題 2 解答

問題の難易度の目安【基礎】★☆☆　【標準】★★☆　【発展】★★★

1 (★☆☆)(偏微分作用素①)

以下の問いに答えよ．

(1) f は C^2 級とする．$\left(3\dfrac{\partial}{\partial x}+2\dfrac{\partial}{\partial y}\right)^2 f$ を f_{xx}, f_{xy}, f_{yy} を用いて表せ．

(2) $\left(3\dfrac{\partial}{\partial x}+2\dfrac{\partial}{\partial y}\right)^2 e^{x+2y}$ を求めよ．

解 (1) f を C^2 級とすると，$f_{xy}=f_{yx}$ すなわち $\dfrac{\partial^2 f}{\partial y\partial x}=\dfrac{\partial^2 f}{\partial x\partial y}$ が成り立つ．したがって，

$$\begin{aligned}\left(3\frac{\partial}{\partial x}+2\frac{\partial}{\partial y}\right)^2 f &= \left(3\frac{\partial}{\partial x}+2\frac{\partial}{\partial y}\right)\left(3\frac{\partial f}{\partial x}+2\frac{\partial f}{\partial y}\right)\\ &= 9\frac{\partial^2 f}{\partial^2 x}+6\frac{\partial^2 f}{\partial x\partial y}+6\frac{\partial^2 f}{\partial y\partial x}+4\frac{\partial^2 f}{\partial^2 y}\\ &= 9\frac{\partial^2 f}{\partial^2 x}+12\frac{\partial^2 f}{\partial y\partial x}+4\frac{\partial^2 f}{\partial^2 y}\\ &= \boldsymbol{9f_{xx}+12f_{xy}+4f_{yy}}.\end{aligned}$$

(2) $f(x,y)=e^{x+2y}$ とおくと，明らかに f は C^2 級であり $f_x=e^{x+2y}$，$f_y=2e^{x+2y}$ より

$$f_{xx}=e^{x+2y},\quad f_{xy}=f_{yx}=2e^{x+2y},\quad f_{yy}=4e^{x+2y}.$$

ゆえに，(1) の結果を用いると

$$\begin{aligned}\left(3\frac{\partial}{\partial x}+2\frac{\partial}{\partial y}\right)^2 e^{x+2y} &= 9e^{x+2y}+12\cdot 2e^{x+2y}+4\cdot 4e^{x+2y}\\ &= \boldsymbol{49e^{x+2y}}.\end{aligned}$$

■

2 (★☆☆)(熱方程式)

$t>0$，$-\infty<x<+\infty$ で定義された関数 $u(t,x)$ に対して

$$u_t(t,x)=u_{xx}(t,x),\quad (t,x)\in(0,\infty)\times\mathbb{R} \tag{H}$$

を (1 次元) **熱方程式**という．**熱核**とよばれる関数

$$p(t,x)=\frac{1}{\sqrt{4\pi t}}e^{-\frac{x^2}{4t}}$$

は熱方程式 (H) を満たすことを示せ.

解 $p(t,x)=\dfrac{1}{\sqrt{4\pi t}}e^{-\frac{x^2}{4t}}$ に対して,

$$p_x=\frac{1}{\sqrt{4\pi t}}\left(-\frac{2x}{4t}\right)e^{-\frac{x^2}{4t}},$$
$$p_{xx}=\frac{1}{\sqrt{4\pi t}}\left[\left(-\frac{2x}{4t}\right)^2-\frac{1}{2t}\right]e^{-\frac{x^2}{4t}}\quad\cdots①$$

であり,

$$\begin{aligned}p_t&=-\frac{1}{2t\sqrt{4\pi t}}e^{-\frac{x^2}{4t}}+\frac{1}{\sqrt{4\pi t}}\left(\frac{x^2}{t^2}\right)e^{-\frac{x^2}{4t}}\\&=\frac{1}{\sqrt{4\pi t}}\left[\frac{x^2}{4t^2}-\frac{1}{2t}\right]e^{-\frac{x^2}{4t}}\quad\cdots②\end{aligned}$$

であるから，①,②より，$p_t=p_{xx}$ を満たすことがわかる．したがって，p は熱方程式 (H) の解である. ■

3 (★★☆)(chain rule)

$f(r)$ は 1 変数 r のなめらかな関数とする．3 変数関数 $u(x,y,z)$ を

$$u(x,y,z):=f\left(\sqrt{x^2+y^2+z^2}\right)$$

で定義する．このとき,

$$(\Delta u=)\ u_{xx}+u_{yy}+u_{zz}=f''(r)+\frac{2}{r}f'(r)$$

となることを示せ.

解 $r=\sqrt{x^2+y^2+z^2}$ とおく．$r^2=x^2+y^2+z^2$ だから，両辺を x で偏微分して，$2rr_x=2r$ すなわち，$r_x=\dfrac{x}{r}\cdots①$．同様に，$r_y=\dfrac{y}{r}\cdots②$，$r_z=\dfrac{z}{r}\cdots③$．ゆえに chain rule により①を用いると

$$u_x(x,y,z)=u_r(x,y,z)r_x=f'(r)\frac{x}{r}.$$

したがって,

$$\begin{aligned}u_{xx}(x,y,z)=(u_r r_x)_r r_x&=u_{rr}(r_x)^2+u_r(r_x)_r r_x\\&=u_{rr}\frac{x^2}{r^2}+u_r\left(\frac{x_r}{r}-\frac{x}{r^2}\right)\frac{x}{r}\\&=f''(r)\frac{x^2}{r^2}+f'(r)\left(\frac{xx_r}{r^2}-\frac{x^2}{r^3}\right).\end{aligned}$$

ここで，再び $r^2 = x^2 + y^2 + z^2$ より両辺を r で微分して，$2r = 2xx_r$，すなわち $xx_r = r$．これを上式に代入して，

$$u_{xx} = f''(r)\frac{x^2}{r^2} + f'(r)\left(\frac{1}{r} - \frac{x^2}{r^3}\right) \quad \cdots ④.$$

全く同様にして

$$u_{yy} = f''(r)\frac{y^2}{r^2} + f'(r)\left(\frac{1}{r} - \frac{y^2}{r^3}\right) \quad \cdots ⑤$$

$$u_{zz} = f''(r)\frac{z^2}{r^2} + f'(r)\left(\frac{1}{r} - \frac{z^2}{r^3}\right) \quad \cdots ⑥.$$

④,⑤,⑥を足し合わせて $x^2 + y^2 + z^2 = r^2$ を用いると

$$\begin{aligned} u_{xx} + u_{yy} + u_{zz} &= f''(r)\frac{x^2+y^2+z^2}{r^2} + f'(r)\left(\frac{3}{r} - \frac{x^2+y^2+z^2}{r^3}\right) \\ &= f''(r) + \frac{2}{r}f'(r). \end{aligned}$$

∎

4 (★★☆)(偏微分作用素②)

$f(x, y, z)$ はなめらかな関数とし，2つの偏微分作用素 Δ, D を f に対して，

$$\Delta f := \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$$

$$Df := x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} + z\frac{\partial f}{\partial z}$$

で定義する．このとき，以下の問いに答えよ．

(1) $\Delta(Df) = D(\Delta f) + 2\Delta f$ を示せ．

(2) $\Delta f = 0$ のとき，$\Delta\left((x^2+y^2+z^2)f\right)$ を f と DF を用いて表せ．

(3) $\Delta f = 0$ のとき，$\Delta\left[\Delta\left((x^2+y^2+z^2)f\right)\right] = 0$ であることを示せ．

解 (1) f はなめらかだから偏微分の順序交換が自由にできる．$Df = x\dfrac{\partial f}{\partial x} + y\dfrac{\partial f}{\partial y} + z\dfrac{\partial f}{\partial z}$ であるから，

$$\begin{aligned} \Delta(Df) &= \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)\left(x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} + z\frac{\partial f}{\partial z}\right) \\ &= \frac{\partial^2}{\partial x^2}\left(x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} + z\frac{\partial f}{\partial z}\right) + \frac{\partial^2}{\partial y^2}\left(x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} + z\frac{\partial f}{\partial z}\right) \\ &\quad + \frac{\partial^2}{\partial z^2}\left(x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} + z\frac{\partial f}{\partial z}\right) \quad \cdots ① \end{aligned}$$

ここで，$\dfrac{\partial}{\partial x}\left(x\dfrac{\partial f}{\partial x}\right)=\dfrac{\partial f}{\partial x}+x\dfrac{\partial^2 f}{\partial x^2}$ だから，

$$\begin{aligned}\frac{\partial^2}{\partial x^2}\left(x\frac{\partial f}{\partial x}\right)&=\frac{\partial f}{\partial x}\left(\frac{\partial f}{\partial x}+x\frac{\partial^2 f}{\partial x^2}\right)\\&=2\frac{\partial^2 f}{\partial x^2}+x\frac{\partial^3 f}{\partial x^3}\\&=2\frac{\partial^2 f}{\partial x^2}+x\frac{\partial}{\partial x}\left(\frac{\partial^2 f}{\partial x^2}\right).\end{aligned}$$

ゆえに，

$$\begin{aligned}\frac{\partial^2}{\partial x^2}\left(x\frac{\partial f}{\partial x}+y\frac{\partial f}{\partial y}+z\frac{\partial f}{\partial z}\right)&=2\frac{\partial^2 f}{\partial x^2}+x\frac{\partial}{\partial x}\left(\frac{\partial^2 f}{\partial x^2}\right)+y\frac{\partial^3 f}{\partial x^2\partial y}+z\frac{\partial^3 f}{\partial x^2\partial z}\\&=2\frac{\partial^2 f}{\partial x^2}+\left(x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}+z\frac{\partial}{\partial z}\right)\left(\frac{\partial^2 f}{\partial x^2}\right)\\&=2\frac{\partial^2 f}{\partial x^2}+D\left(\frac{\partial^2 f}{\partial x^2}\right)\quad\cdots②.\end{aligned}$$

同様に，

$$\frac{\partial^2}{\partial y^2}\left(x\frac{\partial f}{\partial x}+y\frac{\partial f}{\partial y}+z\frac{\partial f}{\partial z}\right)=2\frac{\partial^2 f}{\partial y^2}+D\left(\frac{\partial^2 f}{\partial y^2}\right)\quad\cdots③,$$

$$\frac{\partial^2}{\partial z^2}\left(x\frac{\partial f}{\partial x}+y\frac{\partial f}{\partial y}+z\frac{\partial f}{\partial z}\right)=2\frac{\partial^2 f}{\partial z^2}+D\left(\frac{\partial^2 f}{\partial z^2}\right)\quad\cdots④.$$

②,③,④を①へ代入して

$$\begin{aligned}\Delta(Df)&=2\left(\frac{\partial^2 f}{\partial x^2}+\frac{\partial^2 f}{\partial y^2}+\frac{\partial^2 f}{\partial z^2}\right)+D\left(\frac{\partial^2 f}{\partial x^2}+\frac{\partial^2 f}{\partial y^2}+\frac{\partial^2 f}{\partial z^2}\right)\\&=2\Delta f+D(\Delta f)\end{aligned}$$

となり，証明された.

(2) $\Delta\left((x^2+y^2+z^2)f\right)=\left(\dfrac{\partial^2}{\partial x^2}+\dfrac{\partial^2}{\partial y^2}+\dfrac{\partial^2}{\partial z^2}\right)((x^2+y^2+z^2)f)$ において，

$$\frac{\partial}{\partial x}\left((x^2+y^2+z^2)f\right)=2xf+(x^2+y^2+z^2)\frac{\partial f}{\partial x}$$

だから，

$$\begin{aligned}\frac{\partial^2}{\partial^2 x}\left((x^2+y^2+z^2)f\right)&=\frac{\partial}{\partial x}\left(2xf+(x^2+y^2+z^2)\frac{\partial f}{\partial x}\right)\\&=2f+2x\frac{\partial f}{\partial x}+2x\frac{\partial f}{\partial x}+(x^2+y^2+z^2)\frac{\partial^2 f}{\partial x^2}\\&=2f+4x\frac{\partial f}{\partial x}+(x^2+y^2+z^2)\frac{\partial^2 f}{\partial x^2}.\end{aligned}$$

同様に,

$$\frac{\partial^2}{\partial^2 y}\left((x^2+y^2+z^2)f\right)=2f+4y\frac{\partial f}{\partial y}+(x^2+y^2+z^2)\frac{\partial^2 f}{\partial y^2}$$

$$\frac{\partial^2}{\partial^2 z}\left((x^2+y^2+z^2)f\right)=2f+4z\frac{\partial f}{\partial y}+(x^2+y^2+z^2)\frac{\partial^2 f}{\partial z^2}$$

である．以上の計算から $\Delta f=0$ に注意して

$$\begin{aligned}\Delta\left((x^2+y^2+z^2)f\right)&=6f+4\left(x\frac{\partial f}{\partial x}+x\frac{\partial f}{\partial x}+x\frac{\partial f}{\partial x}\right)\\&\qquad+(x^2+y^2+z^2)\left(\frac{\partial^2 f}{\partial x^2}+\frac{\partial^2 f}{\partial y^2}+\frac{\partial^2 f}{\partial z^2}\right)\\&=6f+4Df+(x^2+y^2+z^2)\Delta f\\&=\boldsymbol{6f+4Df}.\end{aligned}$$

(3) (2) の結果を用いて,

$$\Delta\left[\Delta\left((x^2+y^2+z^2)f\right)\right]=\Delta(6f+4Df)=6\Delta f+4\Delta(Df)\quad\cdots⑤.$$

f は $\Delta f=0$ をみたし，さらに (1) の等式より

$$\Delta(Df)=D(\Delta f)+2\Delta f=0.$$

よってこれらを⑤へ代入すれば，$\Delta\left[\Delta\left((x^2+y^2+z^2)f\right)\right]=0$ となり，証明が完了した.

■