# 正規直交化法　問題1　解答

グラム・シュミットの正規直交化法を用いて, 次の基底から正規直交基底を作れ.

(1) $a_1 = \begin{pmatrix}1\\0\\0\end{pmatrix}, a_2 = \begin{pmatrix}1\\2\\-1\end{pmatrix}, a_3 = \begin{pmatrix}-1\\1\\2\end{pmatrix}$

[解]: はじめに与えられた基底から直交基底をつくる.

$$b_1 = a_1 = \begin{pmatrix}1\\0\\0\end{pmatrix},$$

$$b_2 = a_2 - \frac{(a_2, b_1)}{(b_1, b_1)}b_1 = \begin{pmatrix}1\\2\\-1\end{pmatrix} - \begin{pmatrix}1\\0\\0\end{pmatrix} = \begin{pmatrix}0\\2\\-1\end{pmatrix},$$

$$b_3 = a_3 - \frac{(a_3, b_1)}{(b_1, b_1)}b_1 - \frac{(a_3, b_2)}{(b_2, b_2)}b_2 = \begin{pmatrix}-1\\1\\2\end{pmatrix} - \begin{pmatrix}-1\\0\\0\end{pmatrix} - \begin{pmatrix}0\\0\\0\end{pmatrix} = \begin{pmatrix}0\\1\\2\end{pmatrix}.$$

この直交基底 $b_1, b_2, b_3$ の正規化を行えば正規直交基底が求められる. したがって, 求める正規直交基底 $c_1, c_2, c_3$ は,

$$c_1 = 1\begin{pmatrix}1\\0\\0\end{pmatrix} = \begin{pmatrix}1\\0\\0\end{pmatrix},\ c_2 = \frac{\sqrt{5}}{5}\begin{pmatrix}0\\2\\-1\end{pmatrix} = \begin{pmatrix}0\\\frac{2\sqrt{5}}{5}\\-\frac{\sqrt{5}}{5}\end{pmatrix},\ c_3 = \frac{\sqrt{5}}{5}\begin{pmatrix}0\\1\\2\end{pmatrix} = \begin{pmatrix}0\\\frac{\sqrt{5}}{5}\\\frac{2\sqrt{5}}{5}\end{pmatrix}.$$

(2) $a_1 = \begin{pmatrix}0\\1\\1\end{pmatrix}, a_2 = \begin{pmatrix}1\\0\\1\end{pmatrix}, a_3 = \begin{pmatrix}1\\1\\0\end{pmatrix}$

[解]: はじめに与えられた基底から直交基底をつくる.

$$b_1 = a_1 = \begin{pmatrix}0\\1\\1\end{pmatrix},$$

$$b_2 = a_2 - \frac{(a_2, b_1)}{(b_1, b_1)}b_1 = \begin{pmatrix}1\\0\\1\end{pmatrix} - \begin{pmatrix}0\\\frac{1}{2}\\\frac{1}{2}\end{pmatrix} = \begin{pmatrix}1\\-\frac{1}{2}\\\frac{1}{2}\end{pmatrix},$$

$$b_3 = a_3 - \frac{(a_3, b_1)}{(b_1, b_1)}b_1 - \frac{(a_3, b_2)}{(b_2, b_2)}b_2 = \begin{pmatrix}1\\1\\0\end{pmatrix} - \begin{pmatrix}0\\\frac{1}{2}\\\frac{1}{2}\end{pmatrix} - \begin{pmatrix}\frac{1}{3}\\-\frac{1}{6}\\\frac{1}{6}\end{pmatrix} = \begin{pmatrix}\frac{2}{3}\\\frac{2}{3}\\-\frac{2}{3}\end{pmatrix}.$$

この直交基底 $b_1, b_2, b_3$ の正規化を行えば正規直交基底が求められる. したがって, 求める正規直交基底 $c_1, c_2, c_3$ は,

$$c_1 = \frac{\sqrt{2}}{2}\begin{pmatrix}0\\1\\1\end{pmatrix} = \begin{pmatrix}0\\\frac{\sqrt{2}}{2}\\\frac{\sqrt{2}}{2}\end{pmatrix},\ c_2 = \frac{\sqrt{6}}{3}\begin{pmatrix}1\\-\frac{1}{2}\\\frac{1}{2}\end{pmatrix} = \begin{pmatrix}\frac{\sqrt{6}}{3}\\-\frac{\sqrt{6}}{6}\\\frac{\sqrt{6}}{6}\end{pmatrix},\ c_3 = \frac{\sqrt{3}}{2}\begin{pmatrix}\frac{2}{3}\\\frac{2}{3}\\-\frac{2}{3}\end{pmatrix} = \begin{pmatrix}\frac{\sqrt{3}}{3}\\\frac{\sqrt{3}}{3}\\-\frac{\sqrt{3}}{3}\end{pmatrix}.$$

(3) $a_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, a_2 = \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}, a_3 = \begin{pmatrix} 1 \\ -2 \\ -1 \end{pmatrix}$

[解]: はじめに与えられた基底から直交基底をつくる.

$$b_1 = a_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix},$$

$$b_2 = a_2 - \frac{(a_2, b_1)}{(b_1, b_1)} b_1 = \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix} - \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix},$$

$$b_3 = a_3 - \frac{(a_3, b_1)}{(b_1, b_1)} b_1 - \frac{(a_3, b_2)}{(b_2, b_2)} b_2 = \begin{pmatrix} 1 \\ -2 \\ -1 \end{pmatrix} - \begin{pmatrix} -\frac{2}{3} \\ -\frac{2}{3} \\ -\frac{2}{3} \end{pmatrix} - \begin{pmatrix} \frac{3}{2} \\ -\frac{3}{2} \\ 0 \end{pmatrix} = \begin{pmatrix} \frac{1}{6} \\ \frac{1}{6} \\ -\frac{1}{3} \end{pmatrix}.$$

この直交基底 $b_1, b_2, b_3$ の正規化を行えば正規直交基底が求められる. したがって, 求める正規直交基底 $c_1, c_2, c_3$ は,

$$c_1 = \frac{\sqrt{3}}{3} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} \frac{\sqrt{3}}{3} \\ \frac{\sqrt{3}}{3} \\ \frac{\sqrt{3}}{3} \end{pmatrix},\ c_2 = \frac{\sqrt{2}}{2} \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} = \begin{pmatrix} \frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} \\ 0 \end{pmatrix},\ c_3 = \sqrt{6} \begin{pmatrix} \frac{1}{6} \\ \frac{1}{6} \\ -\frac{1}{3} \end{pmatrix} = \begin{pmatrix} \frac{\sqrt{6}}{6} \\ \frac{\sqrt{6}}{6} \\ -\frac{\sqrt{6}}{3} \end{pmatrix}.$$

(4) $a_1 = \begin{pmatrix} -1 \\ 2 \\ 2 \end{pmatrix}, a_2 = \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}, a_3 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$

[解]: はじめに与えられた基底から直交基底をつくる.

$$b_1 = a_1 = \begin{pmatrix} -1 \\ 2 \\ 2 \end{pmatrix},$$

$$b_2 = a_2 - \frac{(a_2, b_1)}{(b_1, b_1)} b_1 = \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} - \begin{pmatrix} -\frac{2}{3} \\ \frac{4}{3} \\ \frac{4}{3} \end{pmatrix} = \begin{pmatrix} \frac{2}{3} \\ -\frac{1}{3} \\ \frac{2}{3} \end{pmatrix},$$

$$b_3 = a_3 - \frac{(a_3, b_1)}{(b_1, b_1)} b_1 - \frac{(a_3, b_2)}{(b_2, b_2)} b_2 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} - \begin{pmatrix} -\frac{1}{3} \\ \frac{2}{3} \\ \frac{2}{3} \end{pmatrix} - \begin{pmatrix} \frac{2}{3} \\ -\frac{1}{3} \\ \frac{2}{3} \end{pmatrix} = \begin{pmatrix} \frac{2}{3} \\ \frac{2}{3} \\ -\frac{1}{3} \end{pmatrix}.$$

この直交基底 $b_1, b_2, b_3$ の正規化を行えば正規直交基底が求められる. したがって, 求める正規直交基底 $c_1, c_2, c_3$ は,

$$c_1 = \frac{1}{3} \begin{pmatrix} -1 \\ 2 \\ 2 \end{pmatrix} = \begin{pmatrix} -\frac{1}{3} \\ \frac{2}{3} \\ \frac{2}{3} \end{pmatrix},\ c_2 = 1 \begin{pmatrix} \frac{2}{3} \\ -\frac{1}{3} \\ \frac{2}{3} \end{pmatrix} = \begin{pmatrix} \frac{2}{3} \\ -\frac{1}{3} \\ \frac{2}{3} \end{pmatrix},\ c_3 = 1 \begin{pmatrix} \frac{2}{3} \\ \frac{2}{3} \\ -\frac{1}{3} \end{pmatrix} = \begin{pmatrix} \frac{2}{3} \\ \frac{2}{3} \\ -\frac{1}{3} \end{pmatrix}.$$

(5) $a_1 = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, a_2 = \begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix}, a_3 = \begin{pmatrix} 3 \\ 4 \\ 1 \end{pmatrix}$

[解]: はじめに与えられた基底から直交基底をつくる.

$$b_1 = a_1 = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix},$$

$$b_2 = a_2 - \frac{(a_2, b_1)}{(b_1, b_1)} b_1 = \begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix} - \begin{pmatrix} \frac{10}{7} \\ \frac{20}{7} \\ \frac{30}{7} \end{pmatrix} = \begin{pmatrix} \frac{4}{7} \\ \frac{1}{7} \\ -\frac{2}{7} \end{pmatrix},$$

$$b_3 = a_3 - \frac{(a_3, b_1)}{(b_1, b_1)} b_1 - \frac{(a_3, b_2)}{(b_2, b_2)} b_2 = \begin{pmatrix} 3 \\ 4 \\ 1 \end{pmatrix} - \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} - \begin{pmatrix} \frac{8}{3} \\ \frac{2}{3} \\ -\frac{4}{3} \end{pmatrix} = \begin{pmatrix} -\frac{2}{3} \\ \frac{4}{3} \\ -\frac{2}{3} \end{pmatrix}.$$

この直交基底 $b_1, b_2, b_3$ の正規化を行えば正規直交基底が求められる. したがって, 求める正規直交基底 $c_1, c_2, c_3$ は,

$$c_1 = \frac{\sqrt{14}}{14} \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} \frac{\sqrt{14}}{14} \\ \frac{\sqrt{14}}{7} \\ \frac{3\sqrt{14}}{14} \end{pmatrix}, \ c_2 = \frac{\sqrt{21}}{3} \begin{pmatrix} \frac{4}{7} \\ \frac{1}{7} \\ -\frac{2}{7} \end{pmatrix} = \begin{pmatrix} \frac{4\sqrt{21}}{21} \\ \frac{\sqrt{21}}{21} \\ -\frac{2\sqrt{21}}{21} \end{pmatrix}, \ c_3 = \frac{\sqrt{6}}{4} \begin{pmatrix} -\frac{2}{3} \\ \frac{4}{3} \\ -\frac{2}{3} \end{pmatrix} = \begin{pmatrix} -\frac{\sqrt{6}}{6} \\ \frac{\sqrt{6}}{3} \\ -\frac{\sqrt{6}}{6} \end{pmatrix}.$$

(6) $a_1 = \begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix}, a_2 = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, a_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix}, a_4 = \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}$

[解]: はじめに与えられた基底から直交基底をつくる.

$$b_1 = a_1 = \begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix},$$

$$b_2 = a_2 - \frac{(a_2, b_1)}{(b_1, b_1)} b_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} - \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix},$$

$$b_3 = a_3 - \frac{(a_3, b_1)}{(b_1, b_1)} b_1 - \frac{(a_3, b_2)}{(b_2, b_2)} b_2 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix} - \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} - \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix},$$

$$b_4 = a_4 - \frac{(a_4, b_1)}{(b_1, b_1)} b_1 - \frac{(a_4, b_2)}{(b_2, b_2)} b_2 - \frac{(a_4, b_3)}{(b_3, b_3)} b_3 = \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix} - \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} - \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \\ 0 \\ 0 \end{pmatrix} - \begin{pmatrix} 0 \\ 0 \\ \frac{1}{2} \\ \frac{1}{2} \end{pmatrix} = \begin{pmatrix} -\frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \\ -\frac{1}{2} \end{pmatrix}.$$

この直交基底 $b_1, b_2, b_3, b_4$ の正規化を行えば正規直交基底が求められる. したがって, 求め

る正規直交基底 $c_1, c_2, c_3, c_4$ は,

$$c_1 = \frac{1}{2}\begin{pmatrix}1\\-1\\1\\-1\end{pmatrix} = \begin{pmatrix}\frac{1}{2}\\-\frac{1}{2}\\\frac{1}{2}\\-\frac{1}{2}\end{pmatrix}, \qquad c_2 = \frac{\sqrt{2}}{2}\begin{pmatrix}1\\1\\0\\0\end{pmatrix} = \begin{pmatrix}\frac{\sqrt{2}}{2}\\\frac{\sqrt{2}}{2}\\0\\0\end{pmatrix},$$

$$c_3 = \frac{\sqrt{2}}{2}\begin{pmatrix}0\\0\\1\\1\end{pmatrix} = \begin{pmatrix}0\\0\\\frac{\sqrt{2}}{2}\\\frac{\sqrt{2}}{2}\end{pmatrix}, \qquad c_4 = 1\begin{pmatrix}-\frac{1}{2}\\\frac{1}{2}\\\frac{1}{2}\\-\frac{1}{2}\end{pmatrix} = \begin{pmatrix}-\frac{1}{2}\\\frac{1}{2}\\\frac{1}{2}\\-\frac{1}{2}\end{pmatrix}.$$

(7) $a_1 = \begin{pmatrix}1\\3\\0\\0\end{pmatrix}, a_2 = \begin{pmatrix}-1\\1\\0\\0\end{pmatrix}, a_3 = \begin{pmatrix}0\\0\\1\\1\end{pmatrix}, a_4 = \begin{pmatrix}0\\0\\3\\4\end{pmatrix}$

[解]: はじめに与えられた基底から直交基底をつくる.

$$b_1 = a_1 = \begin{pmatrix}1\\3\\0\\0\end{pmatrix},$$

$$b_2 = a_2 - \frac{(a_2, b_1)}{(b_1, b_1)}b_1 = \begin{pmatrix}-1\\1\\0\\0\end{pmatrix} - \begin{pmatrix}\frac{1}{5}\\\frac{3}{5}\\0\\0\end{pmatrix} = \begin{pmatrix}-\frac{6}{5}\\\frac{2}{5}\\0\\0\end{pmatrix},$$

$$b_3 = a_3 - \frac{(a_3, b_1)}{(b_1, b_1)}b_1 - \frac{(a_3, b_2)}{(b_2, b_2)}b_2 = \begin{pmatrix}0\\0\\1\\1\end{pmatrix} - \begin{pmatrix}0\\0\\0\\0\end{pmatrix} - \begin{pmatrix}0\\0\\0\\0\end{pmatrix} = \begin{pmatrix}0\\0\\1\\1\end{pmatrix},$$

$$b_4 = a_4 - \frac{(a_4, b_1)}{(b_1, b_1)}b_1 - \frac{(a_4, b_2)}{(b_2, b_2)}b_2 - \frac{(a_4, b_3)}{(b_3, b_3)}b_3 = \begin{pmatrix}0\\0\\3\\4\end{pmatrix} - \begin{pmatrix}0\\0\\0\\0\end{pmatrix} - \begin{pmatrix}0\\0\\0\\0\end{pmatrix} - \begin{pmatrix}0\\0\\\frac{7}{2}\\\frac{7}{2}\end{pmatrix} = \begin{pmatrix}0\\0\\-\frac{1}{2}\\\frac{1}{2}\end{pmatrix}.$$

この直交基底 $b_1, b_2, b_3, b_4$ の正規化を行えば正規直交基底が求められる. したがって, 求める正規直交基底 $c_1, c_2, c_3, c_4$ は,

$$c_1 = \frac{\sqrt{10}}{10}\begin{pmatrix}1\\3\\0\\0\end{pmatrix} = \begin{pmatrix}\frac{\sqrt{10}}{10}\\\frac{3\sqrt{10}}{10}\\0\\0\end{pmatrix}, \qquad c_2 = \frac{\sqrt{10}}{4}\begin{pmatrix}-\frac{6}{5}\\\frac{2}{5}\\0\\0\end{pmatrix} = \begin{pmatrix}-\frac{3\sqrt{10}}{10}\\\frac{\sqrt{10}}{10}\\0\\0\end{pmatrix},$$

$$c_3 = \frac{\sqrt{2}}{2}\begin{pmatrix}0\\0\\1\\1\end{pmatrix} = \begin{pmatrix}0\\0\\\frac{\sqrt{2}}{2}\\\frac{\sqrt{2}}{2}\end{pmatrix}, \qquad c_4 = \sqrt{2}\begin{pmatrix}0\\0\\-\frac{1}{2}\\\frac{1}{2}\end{pmatrix} = \begin{pmatrix}0\\0\\-\frac{\sqrt{2}}{2}\\\frac{\sqrt{2}}{2}\end{pmatrix}.$$

(8) $a_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, a_2 = \begin{pmatrix} 2 \\ 1 \\ 2 \\ 1 \end{pmatrix}, a_3 = \begin{pmatrix} 0 \\ -2 \\ 1 \\ -1 \end{pmatrix}, a_4 = \begin{pmatrix} 1 \\ -3 \\ 1 \\ -1 \end{pmatrix}$

[解]: はじめに与えられた基底から直交基底をつくる.

$$b_1 = a_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix},$$

$$b_2 = a_2 - \frac{(a_2, b_1)}{(b_1, b_1)} b_1 = \begin{pmatrix} 2 \\ 1 \\ 2 \\ 1 \end{pmatrix} - \begin{pmatrix} \frac{3}{2} \\ \frac{3}{2} \\ \frac{3}{2} \\ \frac{3}{2} \end{pmatrix} = \begin{pmatrix} \frac{1}{2} \\ -\frac{1}{2} \\ \frac{1}{2} \\ -\frac{1}{2} \end{pmatrix},$$

$$b_3 = a_3 - \frac{(a_3, b_1)}{(b_1, b_1)} b_1 - \frac{(a_3, b_2)}{(b_2, b_2)} b_2 = \begin{pmatrix} 0 \\ -2 \\ 1 \\ -1 \end{pmatrix} - \begin{pmatrix} -\frac{1}{2} \\ -\frac{1}{2} \\ -\frac{1}{2} \\ -\frac{1}{2} \end{pmatrix} - \begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix} = \begin{pmatrix} -\frac{1}{2} \\ -\frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \end{pmatrix},$$

$$b_4 = a_4 - \frac{(a_4, b_1)}{(b_1, b_1)} b_1 - \frac{(a_4, b_2)}{(b_2, b_2)} b_2 - \frac{(a_4, b_3)}{(b_3, b_3)} b_3 = \begin{pmatrix} 1 \\ -3 \\ 1 \\ -1 \end{pmatrix} - \begin{pmatrix} -\frac{1}{2} \\ -\frac{1}{2} \\ -\frac{1}{2} \\ -\frac{1}{2} \end{pmatrix} - \begin{pmatrix} \frac{3}{2} \\ -\frac{3}{2} \\ \frac{3}{2} \\ -\frac{3}{2} \end{pmatrix} - \begin{pmatrix} -\frac{1}{2} \\ -\frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \end{pmatrix} = \begin{pmatrix} \frac{1}{2} \\ -\frac{1}{2} \\ -\frac{1}{2} \\ \frac{1}{2} \end{pmatrix}.$$

この直交基底 $b_1, b_2, b_3, b_4$ の正規化を行えば正規直交基底が求められる. したがって, 求める正規直交基底 $c_1, c_2, c_3, c_4$ は,

$$c_1 = \frac{1}{2} \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \end{pmatrix}, \qquad c_2 = 1 \begin{pmatrix} \frac{1}{2} \\ -\frac{1}{2} \\ \frac{1}{2} \\ -\frac{1}{2} \end{pmatrix} = \begin{pmatrix} \frac{1}{2} \\ -\frac{1}{2} \\ \frac{1}{2} \\ -\frac{1}{2} \end{pmatrix},$$

$$c_3 = 1 \begin{pmatrix} -\frac{1}{2} \\ -\frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \end{pmatrix} = \begin{pmatrix} -\frac{1}{2} \\ -\frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \end{pmatrix}, \qquad c_4 = 1 \begin{pmatrix} \frac{1}{2} \\ -\frac{1}{2} \\ -\frac{1}{2} \\ \frac{1}{2} \end{pmatrix} = \begin{pmatrix} \frac{1}{2} \\ -\frac{1}{2} \\ -\frac{1}{2} \\ \frac{1}{2} \end{pmatrix}.$$

(9) $a_1 = \begin{pmatrix} 1 \\ 1 \\ -1 \\ -1 \end{pmatrix}, a_2 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ -1 \end{pmatrix}, a_3 = \begin{pmatrix} 1 \\ -1 \\ 1 \\ 1 \end{pmatrix}, a_4 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$

[解]: はじめに与えられた基底から直交基底をつくる.

$$b_1 = a_1 = \begin{pmatrix} 1 \\ 1 \\ -1 \\ -1 \end{pmatrix},$$

$$b_2 = a_2 - \frac{(a_2, b_1)}{(b_1, b_1)} b_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ -1 \end{pmatrix} - \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \\ -\frac{1}{2} \\ -\frac{1}{2} \end{pmatrix} = \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \\ \frac{3}{2} \\ -\frac{1}{2} \end{pmatrix},$$

$$b_3 = a_3 - \frac{(a_3, b_1)}{(b_1, b_1)} b_1 - \frac{(a_3, b_2)}{(b_2, b_2)} b_2 = \begin{pmatrix} 1 \\ -1 \\ 1 \\ 1 \end{pmatrix} - \begin{pmatrix} -\frac{1}{2} \\ -\frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \end{pmatrix} - \begin{pmatrix} \frac{1}{6} \\ \frac{1}{6} \\ \frac{1}{2} \\ -\frac{1}{6} \end{pmatrix} = \begin{pmatrix} \frac{4}{3} \\ -\frac{2}{3} \\ 0 \\ \frac{2}{3} \end{pmatrix},$$

$$b_4 = a_4 - \frac{(a_4, b_1)}{(b_1, b_1)} b_1 - \frac{(a_4, b_2)}{(b_2, b_2)} b_2 - \frac{(a_4, b_3)}{(b_3, b_3)} b_3 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} - \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} - \begin{pmatrix} \frac{1}{3} \\ \frac{1}{3} \\ 1 \\ -\frac{1}{3} \end{pmatrix} - \begin{pmatrix} \frac{2}{3} \\ -\frac{1}{3} \\ 0 \\ \frac{1}{3} \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix}.$$

この直交基底 $b_1, b_2, b_3, b_4$ の正規化を行えば正規直交基底が求められる. したがって, 求める正規直交基底 $c_1, c_2, c_3, c_4$ は,

$$c_1 = \frac{1}{2} \begin{pmatrix} 1 \\ 1 \\ -1 \\ -1 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \\ -\frac{1}{2} \\ -\frac{1}{2} \end{pmatrix}, \qquad c_2 = \frac{\sqrt{3}}{3} \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \\ \frac{3}{2} \\ -\frac{1}{2} \end{pmatrix} = \begin{pmatrix} \frac{\sqrt{3}}{6} \\ \frac{\sqrt{3}}{6} \\ \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{6} \end{pmatrix},$$

$$c_3 = \frac{\sqrt{6}}{4} \begin{pmatrix} \frac{4}{3} \\ -\frac{2}{3} \\ 0 \\ \frac{2}{3} \end{pmatrix} = \begin{pmatrix} \frac{\sqrt{6}}{3} \\ -\frac{\sqrt{6}}{6} \\ 0 \\ \frac{\sqrt{6}}{6} \end{pmatrix}, \qquad c_4 = \frac{\sqrt{2}}{2} \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ \frac{\sqrt{2}}{2} \\ 0 \\ \frac{\sqrt{2}}{2} \end{pmatrix}.$$